MATHEMATICS

FILTER SPACES DETERMINED BY RELATIONS. II

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§ 6. *Constructions commuting with M*

The most important construction with which M commutes is the formation of the product of a family of spaces and relations. Not only does it commute but it does so in a natural way as stated in

Theorem 6. Let Ω be an index set and assume that $f_\alpha : E_\alpha \to E_\alpha'$ is a $(\varrho_\alpha, \varrho_\alpha')$-map for each $\alpha \in \Omega$. Then there are homeomorphisms h_Ω and h_Ω' such that the diagram

$$\begin{array}{ccc} M(\prod_\alpha \varrho_\alpha) & \xrightarrow{M(\prod_\alpha f_\alpha)} & M(\prod_\alpha \varrho_\alpha') \\ \Big\downarrow h_\Omega & & \Big\downarrow h_\Omega' \\ \prod_\alpha M(\varrho_\alpha) & \xrightarrow{\prod_\alpha M(f_\alpha)} & \prod_\alpha M(\varrho_\alpha') \end{array}$$

is commutative.

Proof: It can be assumed that each $E_\alpha \neq \phi$ since the theorem is trivial otherwise. If $\mathscr{F}$ is a $\prod_\alpha \varrho_\alpha$-filter then $\pi_\alpha \mathscr{F}$ is a ϱ_α-filter, where π_α is the projection of $\prod_\alpha E_\alpha$ on E_α. Conversely if for each $\alpha \in \Omega$, $\mathscr{F}_\alpha$ is a ϱ_α-filter then $\prod_\alpha \mathscr{F}_\alpha$ (the filter which has as a basis the sets $\prod_\alpha A_\alpha$ where $A_\alpha = E_\alpha$ for all but a finite number of indices and for those $A_\alpha \in \mathscr{F}_\alpha$) is a $\prod_\alpha \varrho_\alpha$-filter.

Obviously $\pi_\alpha(\prod_\alpha \mathscr{F}_\alpha) = \mathscr{F}_\alpha$. In addition $\mathscr{F} = \prod_\alpha (\pi_\alpha \mathscr{F})$ if $\mathscr{F}$ is a $\prod_\alpha \varrho_\alpha$-filter since $\mathscr{F}$ then has a base of cuboids.

Let $\mathscr{M}$ be a maximal $\prod_\alpha \varrho_\alpha$-filter. Then each $\pi_\alpha \mathscr{M}$ is a maximal ϱ_α-filter. Assume $\mathscr{M}_\alpha \supset \pi_\alpha \mathscr{M}$. Then $\prod_\alpha \mathscr{M}_\alpha \supset \prod_\alpha (\pi_\alpha \mathscr{M}) = \mathscr{M}$. The maximality of $\mathscr{M}$ implies that $\prod_\alpha \mathscr{M}_\alpha = \mathscr{M}$ and hence that $\mathscr{M}_\alpha = \pi_\alpha \mathscr{M}$.

Similarly if each $\mathscr{M}_\alpha$ is a maximal ϱ_α-filter then $\prod_\alpha \mathscr{M}_\alpha$ is a maximal $\prod_\alpha \varrho_\alpha$-filter.

Define $h_\Omega \mathscr{M} = (\pi_\alpha \mathscr{M})_{\alpha \in \Omega}$ and $h_\Omega' \mathscr{M}' = (\pi_\alpha \mathscr{M}')_{\alpha \in \Omega}$. Clearly these functions are $1-1$ and onto. In addition $h_\Omega' \circ M(\prod_\alpha f_\alpha) = [\prod_\alpha M(f_\alpha)] \circ h_\Omega$ since $(\prod_\alpha f_\alpha) \mathscr{M} \supset \mathscr{M}'$ iff $f_\alpha(\pi_\alpha \mathscr{M}) \supset \pi_\alpha' \mathscr{M}'$ for each $\alpha \in \Omega$.

The basis for the product topology on $\prod_\alpha E_\alpha$ defines a basis for the topology of $M(\prod_\alpha \varrho_\alpha)$. Let $O = \prod_\alpha O_\alpha$, where $O_\alpha \in \mathscr{O}_\alpha$ for each α and $O_\alpha = E_\alpha$ for all but a finite number of indices. Since $\mathscr{M} = \prod_\alpha (\pi_\alpha \mathscr{M})$ it is clear that $O \in \mathscr{M}$ iff $O_\alpha \in \pi_\alpha \mathscr{M}$ for each $\alpha \in \Omega$. Hence $h_\Omega O^* = \prod_\alpha O_\alpha^*$ and $h_\Omega^{-1} \prod_\alpha O_\alpha^* = O^*$. This shows that h_Ω and h_Ω' are homeomorphisms.

Example 11. $M(\prod^n \gamma_2)$ is homeomorphic to $\prod^n M(\gamma_2)$ which is homeomorphic to R^n. $M(\prod^n \gamma_4)$ is homeomorphic to $\bar{\mathsf{R}}^n$ i.e. the closed unit cube in R^n. The spaces $M(\prod_{\alpha=1}^{n} \gamma_{i(\alpha)})$ where $i(\alpha) = 2, \pm 3$ or 4 are homeomorphic to the subspaces of the closed unit cube obtained by omitting faces. $M(\gamma_0 \times (-\gamma_0)) = (-\infty, +\infty)$. Consequently $(D, \mathsf{R} \times \mathsf{R}, \gamma_0 \times (-\gamma_0))$ does not satisfy (S) – where D is image of R under the diagonal map $dx = (x, x)$-since $d^{-1} (\gamma_0 \times (-\gamma_0)) = \gamma_4$.

Let $f : E \to E'$ be a (ϱ, ϱ')-map and assume that ϱ, ϱ' satisfy R_4. Define $g = g_f : E \to E \times E'$ by $gx = (x, fx)$. Then gE is the graph $\Gamma = \Gamma_f$ of f.

Proposition 5. $g^{-1}(\varrho \times \varrho') = \varrho$ and the following assertions are equivalent:

(1) the domain of $M(f)$ is $M(\varrho)$; and

(2) $(\Gamma, E \times E', \varrho \times \varrho')$ satisfies (S).

Proof: $g^{-1}(O \times O') = (O \cap f^{-1} O')$. Since ϱ satisfies R_4 it follows that $g^{-1} (\varrho \times \varrho') = \varrho$.

The function $f = \pi' \circ i \circ g$ where $i : \Gamma \to E \times E'$ is the inclusion. $M(g) : M(\varrho) \to M(\varrho \times \varrho' \mid \Gamma)$ is a homeomorphism by theorem 4. (2) is equivalent to the assertion that the domain of $M(i)$ is $M(\varrho \times \varrho' \mid \Gamma)$. When this is so $M(f) = M(\pi') \circ M(i) \circ M(g)$. Consequently (2) implies (1).

Assume (1). Since $g^{-1}(\mathscr{M} \times M(f)\mathscr{M}) = \mathscr{M} \cap f^{-1}\mathscr{M} \subset \mathscr{M}$ it follows that $M(g)\mathscr{M} = \mathscr{M} \times M(f)\mathscr{M} \mid \Gamma$. In other words every maximal $\varrho \times \varrho' \mid \Gamma$-filter is of the form $\mathscr{M} \times M(f)\mathscr{M} \mid \Gamma$. The corollary to theorem 4 shows that (2) holds.

Corollary. Let $f : E \to E'$ be a (ϱ, ϱ')-map and assume ϱ, ϱ' satisfy R_4. If the domain of $M(f)$ is $M(\varrho)$ then M preserves the graph of f. Specifically $M(i) : M(\varrho \times \varrho' \mid \Gamma_f) \to \Gamma_{M(f)}$ is a homeomorphism.

Proof: $M(i) (\mathscr{M} \times M(f)\mathscr{M} \mid \Gamma) = \mathscr{M} \times M(f)\mathscr{M}$.

Example 12. Consider the function $tx=(x, 1/x)$ defined on $\mathsf{R}-\{0\}$ and the relation $\gamma_2 \times \gamma_2$ on R^2. The domain of $M(t)$ is not $M(\varrho)$, $\varrho=t^{-1}(\gamma_2 \times \gamma_2)$ since $(\{(x, y) \mid xy=1\}, \mathsf{R}^2, \gamma_2 \times \gamma_2)$ does not satisfy (S). If this hyperbola A satisfies (S) $M(\gamma_2 \times \gamma_2)=\mathsf{R}^2$ implies that every $\gamma_2 \times \gamma_2 \mid A$-filter has non-void adherence. The trace of $\mathscr{V}(0) \times \mathsf{R}$ on A obviously has void adherence. For the same reason the subsets B of R^6 and E of R^4 do not satisfy (S) where

$$B=\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \mid x_1=x_3=x_5;\ x_2=x_4=x_6;\ x_7=0;\ \text{and } x_1 x_2=1\} \text{ and}$$
$$E=B \cap \mathsf{R}^4 \ (\mathsf{R}^4 \text{ being identified with } x_5=x_6=x_7=0).$$

Two constructions involving products are the formation of direct limits and pullbacks. M commutes with neither.

For example, consider the two functions $f_1, f_2 : \mathsf{R}^2 \to \mathsf{R}^3$ defined by $f_1(x, y)=(x, y, 0)$ and $f_2(x, y)=(x, y, 1-xy)$. If M is to commute with direct limits (pullbacks) it is necessary that $(B, \mathsf{R}^7, \prod^7\gamma_2)$ $((E, \mathsf{R}^4, \prod^4\gamma_2))$ satisfy (S).

An obvious corollary of the product theorem is that (S) is inherited by products i.e. if $(A_\alpha, E_\alpha, \varrho_\alpha)$ satisfies (S) for $\alpha \in \Omega$ $(\prod\limits_\alpha A_\alpha, \prod\limits_\alpha E_\alpha, \prod\limits_\alpha \varrho_\alpha)$ satisfies the condition. Hence if the function t of example 12 is factored as $t=(e \times j) \circ d$ where $dx=(x, x)$ and $ex=x$, $jx=1/x$, $x \neq 0$ it follows that $M(d)$ is not an embedding of $M(\varrho)$ in $M(e^{-1}(\gamma_2) \times j^{-1}(\gamma_2))$ i.e. the domain of $M(d)$ is not $M(\varrho)$.

However if $d_\Omega : E \to \prod\limits_\alpha E$ is the diagonal map $d_\Omega x=(x)_\alpha$ and $\varrho_\alpha=\varrho$ for each $\alpha \in \Omega$ satisfies R_4 then $M(d_\Omega)$ is an embedding. This is an immediate consequence of the fact that $h_\Omega \circ M(d_\Omega)\mathscr{M}=(\mathscr{M})_\alpha$. In other words $M(d_\Omega)$ is essentially a diagonal map.

The diagonal of $\prod\limits_\alpha M(\varrho)=\{(\mathscr{M}_\alpha)_\alpha \mid \{\mathscr{M}_\alpha \mid \alpha \in \Omega\}$ generate a filter$\}$. This suggests the following extension in case ϱ_α varies with α: the diagonal of $\prod\limits_\alpha M(\varrho_\alpha)=\{(\mathscr{M}_\alpha)_\alpha \mid \{\mathscr{M}_\alpha \mid \alpha \in \Omega\}$ generate a filter$\}$. Clearly $(\mathscr{M}_\alpha)_\alpha$ is in the diagonal iff it is in the range of $h_\Omega \circ M(d_\Omega)$.

In general the diagonal of $\prod\limits_\alpha M(\varrho_\alpha)$ is the not homeomorphic to $M(\bigwedge\limits_\alpha \varrho_\alpha)$ under $h_\Omega \circ M(d_\Omega)$. However if $D(\bigwedge\limits_\alpha \varrho_\alpha)$ is the domain of $M(d_\Omega)$ then as a subspace $D(\bigwedge\limits_\alpha \varrho_\alpha)$ is clearly homeomorphic to the diagonal. This raises the following problem: given $(\varrho_\alpha)_\alpha$ all satisfying R_4 is there a relation ϱ such that the identity map of E induces a homeomorphism of $M(\varrho)$ with $D(\bigwedge\limits_\alpha \varrho_\alpha)$?

If $\Omega=\{\alpha_1, \ldots, \alpha_n\}$ let $\mathop{\triangle}\limits_\Omega \varrho_\alpha$ be the relation: $O \mathop{\triangle}\limits_\Omega \varrho_\alpha P$ if $O=\bigcap\limits_{i=1}^n O_i$, $P=\bigcap\limits_{i=1}^n P_i$ and $O_i\, \varrho_{\alpha_i} P_i$, $i=1, \ldots, n$. Clearly $\mathop{\triangle}\limits_\Omega \varrho_\alpha \subset \bigwedge\limits_\alpha \varrho_\alpha$. If Ω is arbitrary and $\Phi \subset \Omega$ is finite the subrelation $\underset{\Phi}{\triangle}\, \varrho_\alpha$ of $\underset{\alpha}{\wedge}\, \varrho_\alpha$ is defined in the obvious way.

Proposition 6. Assume each ϱ_α satisfies R_4. The following assertions about a filter $\mathscr{F}$ are equivalent:

(1) $A \in \mathscr{F}$ and $\Phi_1 \subset \Omega \Rightarrow$ there is $\Phi_2 \supset \Phi_1$ and $O, P \in \mathscr{F}$ with $O \underset{\Phi_2}{\triangle}\, \varrho_\alpha P$ and $P \subset A$;

(2) $\mathscr{F}$ contains a ϱ_α-filter for each $\alpha \in \Omega$ and these subfilters generate $\mathscr{F}$.

Proof: Let $\alpha \in \Omega$ and let $\Phi_1 = \{\alpha\}$. If $A \in \mathscr{F}$ there exists $\Phi_2 \subset \Omega$ containing α with $O \underset{\Phi_2}{\triangle}\, \varrho_\alpha P$, $O \in \mathscr{F}$ and $P \subset A$. Using Φ_2 as Φ_1 and O as A and so on it is clear that $\mathscr{F}$ contains a ϱ_α-filter.

Since all the relations satisfy R_4, $\mathscr{F}$ contains a largest ϱ_α-filter $\mathscr{F}_\alpha$. It is clear that the $\mathscr{F}_\alpha$ generate $\mathscr{F}$.

Assume (2) holds. Then if $A \in \mathscr{F}$ there is a finite subset $\Phi \subset \Omega$ with $\Phi = \{\alpha_1, \ldots, \alpha_n\}$ and $A \supset \bigcap_{i=1}^{n} P_i$ with $O_i\, \varrho_{\alpha_i} P_i$, $i = 1, \ldots, n$ and $\bigcap_{i=1}^{n} O_i \in \mathscr{F}$. Hence if $\Phi_1 \subset \Omega$ is finite and $\Phi_2 = \Phi \cup \Phi_1$ the choice of $(O_\alpha, P_\alpha) \in \varrho_\alpha$, $\alpha \in \Phi_1$ and $O_\alpha \in \mathscr{F}_\alpha$ shows that there exists $O \in \mathscr{F}$, $P \subset A$ with $O \underset{\Phi_2}{\triangle}\, \varrho_\alpha P$.

Corollary. If $\varrho_\alpha = f^{-1}(\gamma_{i(\alpha)})$ $i(\alpha) = 2, \pm 3,$ or 4 where $\alpha \in \Omega$ the filters in $D(\underset{\alpha}{\wedge}\, \varrho_\alpha)$ are precisely those maximal with respect to (1).

Proof: If $\mathscr{F}$ is maximal among the filters satisfying (1), then each $\mathscr{F}_\alpha$ is a maximal ϱ_α-filter. Pick $\alpha \in \Omega$. If the filter $f_\alpha \mathscr{F}$ has non-void adherence and λ is in this set then $\mathscr{F}$ and $(f_\alpha^{-1}(\lambda - 1/n, \lambda + 1/n))_n$ generate a filter satisfying (1). Hence $\mathscr{F}_\alpha \supset f_\alpha^{-1} \mathscr{V}(\lambda)$ and so $\mathscr{F}_\alpha$ is a maximal ϱ_α-filter. If on the other hand $f_\alpha \mathscr{F}$ has void adherence then $i(\alpha) = \pm 3$ or 4 and by the same reasoning $\mathscr{F}_\alpha = f^{-1} \mathscr{V}(+\infty)$ or $f^{-1} \mathscr{V}(-\infty)$.

Assume Ω is countable i.e. $\Omega = \mathbf{N}$. Let $[n] = \{1, \ldots, n\}$ and define the relation $\underset{n}{\triangle}\, \varrho_n$ as follows: $O \underset{n}{\triangle}\, \varrho_n P$ if there is an integer m and sets $Q \subset R$ with $O \underset{[m+1]}{\triangle}\, \varrho_n Q$ and $R \underset{[m]}{\triangle}\, \varrho_n P$. Then a filter $\mathscr{F}$ is a $\underset{n}{\triangle}\, \varrho_n$-filter iff it contains a ϱ_n-filter for each n and is generated by them.

Proposition 7. $M(e)^{-1} : M(\underset{n}{\triangle}\, \varrho_n) \to D(\underset{n}{\wedge}\, \varrho_n)$ is a homeomorphism if $\varrho_n = f_n^{-1}\,(\gamma_{i(n)})$ where $i(n) = 2, \pm 3,$ or 4.

Proof: $\underset{n}{\triangle}\, \varrho_n$ is a subrelation of $\underset{n}{\wedge}\, \varrho_n$ and so $M(e)^{-1} : M(\underset{n}{\triangle}\, \varrho_n) \to M(\underset{n}{\wedge}\, \varrho_n)$. The characterization of $\underset{n}{\triangle}\, \varrho_n$-filters and the corollary to proposition 6 show that $M(e)^{-1}$ is an embedding with image $D(\underset{n}{\wedge}\, \varrho_n)$.

The functor M commutes naturally with sums.

Theorem 7. Let Ω be an index set and assume that $f_\alpha : E_\alpha \to E'_\alpha$, is a $(\varrho_\alpha, \varrho_\alpha')$-map for each $\alpha \in \Omega$. Then there are homeomorphisms g_Ω and g_Ω' such that the following diagram

$$\begin{array}{ccc} M(\sum_\alpha \varrho_\alpha) & \xrightarrow{M(\sum_\alpha f_\alpha)} & M(\sum_\alpha \varrho_\alpha') \\ \uparrow g_\Omega & & \uparrow g_\Omega' \\ \sum_\alpha M(\varrho_\alpha) & \xrightarrow{\sum_\alpha M(f_\alpha)} & \sum_\alpha M(\varrho_\alpha') \end{array}$$

is commutative.

Proof: A filter $\mathscr{F}$ on $\sum_\alpha E_\alpha$ contains at most one E_{α_0} and if $E_{\alpha_0} \in \mathscr{F}$ then $\mathscr{F} \mid E_{\alpha_0}$ is obviously a filter on E_{α_0}. Let $\mathscr{F}_\alpha$ be a ϱ_α-filter on E_α and let $\{\mathscr{F}_\alpha\}$ be the filter it generates on $\sum_\alpha \varrho_\alpha$. Then $\{\mathscr{F}_\alpha\}$ is a $\sum_\alpha \varrho_\alpha$-filter.

If $\mathscr{M}_\alpha$ is a maximal ϱ_α-filter on E_α then $\{\mathscr{M}_\alpha\}$ is a maximal $\sum_\alpha \varrho_\alpha$-filter. Assume $\mathscr{M} \supset \{\mathscr{M}_\alpha\}$. Then $E_\alpha \in \mathscr{M}$ and so $\mathscr{M} \mid E_\alpha = \mathscr{M}_\alpha$ by the maximality of $\mathscr{M}_\alpha$. Since $\mathscr{M} \subset \{\mathscr{M} \mid E_\alpha\}$ it follows that $\mathscr{M} = \mathscr{M}_\alpha$.

Similarly if $E_\alpha \in \mathscr{M}$ — a maximal $\sum_\alpha \varrho_\alpha$-filter — then $\mathscr{M} \mid E_\alpha$ is a maximal ϱ_α-filter.

Every maximal $\sum_\alpha \varrho_\alpha$-filter $\mathscr{M}$ is of the form $\{\mathscr{M}_\alpha\}$ for some α. Since $\mathscr{M}$ is a $\sum_\alpha \varrho_\alpha$-filter it contains a set O of the form $O = \bigcup_\alpha O_\alpha$. Consequently for one of these α, say α_1, $\mathscr{M} \mid E_{\alpha_1}$ is a ϱ_1-filter. Let $\mathscr{M}_{\alpha_1} \supset \mathscr{M} \mid E_{\alpha_1}$ be a maximal ϱ_1-filter. Then $\{\mathscr{M}_{\alpha_1}\} \supset \mathscr{M}$ and so by the maximality of $\mathscr{M}$ they coincide.

Define $g_\Omega \mathscr{M}_\alpha = \{\mathscr{M}_\alpha\}$. Then $g_\Omega' \circ \sum_\alpha M(f_\alpha) = M(\sum_\alpha f_\alpha) \circ g_\Omega$ as $f_\alpha \mathscr{M}_\alpha \supset \mathscr{M}_\alpha'$ iff $(\sum_\alpha f_\alpha) \{\mathscr{M}_\alpha\} \supset \{\mathscr{M}_\alpha'\}$ (remember $f\mathscr{F}$ is the filter generated by the sets fA, $A \in \mathscr{F}$). The fact that g_Ω is a homeomorphism follows immediately from the observation that $O \in \{\mathscr{M}_\alpha\}$ iff $O \cap E_\alpha \in \mathscr{M}_\alpha$.

Since products and subspaces are dual to sums and quotients it is natural to consider M and quotients. Let $(E, \mathscr{O}, \varrho)$ be a triple and let r be an equivalence relation on E. Denote by ϱ_r the subset of ϱ equal to $\{(O, P) \in \varrho \mid r[O] = O, r[P] = P\}$ and by ϱ/r the finest relation on E/r for which $\pi : E \to E/r$ is an admissable map.

It is clear that $\pi^{-1}(\varrho/r) = \varrho_r$ and hence (when ϱ satisfies R_4) that $M(\pi) : M(\varrho_r) \to M(\varrho/r)$ is a homeomorphism. For relations ϱ satisfying R_4 this reduces the study of quotients to the consideration of $\varrho \supset \varrho_r$ and $M(e) : M(\varrho) \to M(\varrho_r)$.

If r is an open equivalence relation it will be said to be compatible with ϱ if $O \, \varrho \, P$ implies $r[O] \, \varrho \, r[P]$. When r is compatible and ϱ satisfies R_4 every maximal ϱ-filter $\mathscr{M}$ contains a maximal ϱ_r-filter. Let $r[\mathscr{M}]$ be the filter generated by the sets $r[A]$, $A \in \mathscr{M}$. It is the largest ϱ_r-filter contained in ϱ. Assume it is not maximal and let $\mathscr{N}$ be a maximal ϱ_r-filter containing it. Since ϱ satisfies R_4 there are disjoint open sets O, P with $O \in \mathscr{M}$ and $P \in \mathscr{N}$. As P can be assumed to be r-saturated $O \cap P = \phi$ implies $r[O] \cap P = \phi$. This is a contradiction. Consequently $r[\mathscr{M}]$ is the unique maximal ϱ_r-filter contained in $\mathscr{M}$ i.e. $r[\mathscr{M}] = M(e)\mathscr{M}$.

The function $M(e) : M(\varrho) \to M(\varrho_r)$ is clearly onto and so $M(\varrho_r)$ can be considered as the quotient of the set $M(\varrho)$ by the equivalence relation r_M defined by $r[\mathscr{M}_1] = r[\mathscr{M}_2]$. Since $M(e)$ is continuous the topology of $M(\varrho_r)$ is coarser than the quotient topology. It coincides with the quotient topology iff every r_M-saturated open subset of $M(\varrho)$ is a union of sets of the form $(r[O])^*$. This is an immediate consequence of the fact that an r-saturated open set $r[O]$ is in $\mathscr{M}$ iff it is in $r[\mathscr{M}]$.

Theorem 8. Let ϱ be a relation satisfying R_4 on a topology $\mathscr{O}$ for E and let r be a compatible equivalence relation on E. Assume further that $O \,\varrho\, P \Rightarrow (r\,[O])^* \subset r_M[P^*]$. Then there is a homeomorphism $k : M(\varrho)/r_M \to M(\varrho/r)$ such that $k \circ \pi_M = M(\pi)$, where π_M is the canonical map.

Proof: From what has been said it is clear that a continuous bijection k exists with these properties. If $O \,\varrho\, P \Rightarrow (r[O])^* \subset r_M[P^*]$ then it is clear that every r_M-saturated open subset of $M(\varrho)$ is a union of sets of the form $(r[O])^*$. Hence k is a homeomorphism.

Sums and quotients are used to define inductive limits and pushouts. M does not commute with inductive limits. For example consider the trivial system consisting of the identity function $e : \mathsf{R} \to \mathsf{R}$ as a (γ_1, γ_0)-map. The inductive limit of e is (R, γ_0) together with the usual functions. However the inductive limit of $M(e)$ is not defined since the domain of $M(e)$ is not $M(\gamma_1)$.

A consequence of theorems 7 and 8 is that M commutes with certain types of pushouts. Let E be a topological space and let $f_j : E \to E_j$, $j = 1, 2$ be two continuous functions. They define an equivalence relation $r = r\,(f_1, f_2)$ on $E_1 + E_2$ which is generated by $\{(f_1x, f_2x) | x \in E\}$. The quotient space $E_1 + E_2/r(f_1, f_2)$ and the functions $\pi \circ i_j$ (where i_j is the inclusion of E_j in $E_1 + E_2$ and π is the canonical map) is called the pushout of f_1 and f_2. Any other space and pair of functions with the same mapping properties is also called the pushout of f_1 and f_2.

Assume that relations ϱ, ϱ_1 and ϱ_2 are defined on the appropriate topologies and the functions f_j are admissable. Then the pushout consists of the topological pushout and the relation $\varrho_1 + \varrho_2/r(f_1, f_2)^*$.

Let X be the subset of $(E_1 + E_2) \times (E_1 + E_2)$ consisting of the diagonal and $\{(f_kx, f_jx) \,|\, x \in E$ and $k, j = 1$ or $2\}$. Then $r = \bigcup_{n \geqslant 1} X^n$ and $r[A] = \bigcup_{n \geqslant 1} X^n[A]$. For any subset A_1 of E_1 $X[A_1] = A_1 + f_2 f_1^{-1} A_1$ and similarly $X[A_2] = A_2 + {} + f_1 f_2^{-1} A_2$ if $A_2 \subset E_2$. Hence if the functions f_j are open $X[O]$ is open when O is open and so r is open. If in addition $f_j(\varrho) \subset \varrho_j$ and ϱ_j satisfies R_3 it follows that $O(\varrho_1+\varrho_2)P \Rightarrow X[O]\,(\varrho_1+\varrho_2)X[P]$.

Assume $r=r(f_1, f_2)$ has the following property: for any open set O there is an integer $n=n(O)$ such that $r\,[O]=X^n[O]$. Then the compatibility of X with $\varrho_1+\varrho_2$ implies that r is compatible. In particular this is the case if $f_1x=f_1y \Rightarrow f_2x=f_2y$ since then $f_2^{-1}\,f_2\,f_1^{-1}=f_1^{-1}$ (in this case $n=3$ will work for any set).

Proposition 8. Let $j=1, 2$. Assume ϱ_j satisfies R_3, R_4, R_7 and ϱ satisfies R_4, R_7. Let $f_j : E \to E_j$ be a (ϱ, ϱ_j)-map such that f_j is open and $f_j(\varrho) \subset \varrho_j$. Assume that for any open set O there exists $n=n\,(O)$ such that $r[O]=X^n[O]$, $r=r\,(f_1, f_2)$. Then

$$M(\varrho_1+\varrho_2/r(f_1, f_2) = M(\varrho_1)+M(\varrho_2)/r(M(f_1), M(f_2)).$$

Proof: The relation $M(f_1)$ is a continuous function defined on $M(\varrho)$ since $M(f_1)\mathscr{M}=f_1\mathscr{M}$. Clearly $f_1\mathscr{M}$ is a ϱ_1-filter. Let $\mathscr{M}_1 \supset f_1\mathscr{M}$. Then since ϱ satisfies $R_4, \mathscr{M} \supset f_1^{-1}\mathscr{M}_1 \supset f_1^{-1}\,f_1\mathscr{M}$. Consequently $f_1\mathscr{M} \supset f_1\,f_1^{-1}\mathscr{M}_1 \supset \mathscr{M}_1$. Similarly $M(f_2)$ is a function on $M(\varrho)$. Continuity follows from proposition 4 as ϱ_1 and ϱ_2 satisfy R_4 and R_7.

Identify $M(\varrho_1)+M(\varrho_2)$ with $M(\varrho_1+\varrho_2)$ by the homeomorphism defined in theorem 7. Then $r\,(M(f_1), M(f_2))=(r(f_1, f_2))_M$. Letting $M(f_j)\mathscr{M}$ also denote the filter it generates on E_1+E_2 it is sufficient to show that $r[M(f_1)\mathscr{M}]=r[M(f_2)\mathscr{M}]$ for each $\mathscr{M} \in$ in $M(\varrho)$. This is so because $M(\pi \circ i_j)=$ $=M(\pi) \circ M(i_j)$ is a function and hence $M(\pi \circ i_j \circ f_j)=M(\pi \circ i_j) \circ M(f_j)$. The filter $r\,[M(f_1)\mathscr{M}]$ is generated by the sets $r\,[f_1\,A]$, $A \in \mathscr{M}$. Since $r\,[f_1\,A] \supset f_2\,A$ it follows by symmetry that $r\,[f_1\,A]=r\,[f_2\,A]$. Consequently $r\,[M(f_1)\mathscr{M}]=r\,[M(f_2)\mathscr{M}]$.

To complete the proof it is sufficient to show that the conditions of theorem 8 are satisfied for $\varrho_1+\varrho_2$ and $r=r\,(f_1, f_2)$. Since each $O=O_1+O_2$ it is enough to prove that $O_1\,\varrho_1\,P_1 \Rightarrow (r[O_1])^* \subset r_M[P_1{}^*]$.

Let Y be the subset of $[M(\varrho_1)+M(\varrho_2)] \times [M(\varrho_1)+M(\varrho_2)]$ consisting of the diagonal and $\{(M(f_k)\mathscr{M}, M(f_j)\mathscr{M}) \mid \mathscr{M} \in M(\varrho)$ and $k, j=1$ or $2\}$. Then $r_M = \bigcup\limits_{n\geqslant 1} Y^n$.

Lemma. $O_1\,\varrho_1\,P_1$ and $f_2\,f_1^{-1}\,O_1 \in \mathscr{M}_2 \Rightarrow$ there is an $\mathscr{M}$ with $M(f_2)\mathscr{M}=$ $=f_2\mathscr{M}=\mathscr{M}_2$ and $f_1^{-1}\,P_1 \in \mathscr{M}$. In other words $(f_2\,f_1^{-1}\,O_1)^* \subset M(f_2)\,(f_1^{-1}\,P_1)^*$. Similarly $(f_1\,f_2^{-1}\,O_2)^* \subset M(f_1)\,(f_2^{-1}P_2)^*$ if $O_2\,\varrho_2\,P_2$.

Proof: The set $f_1^{-1}\,O_1$ has non-void intersection with each $f_2^{-1}A_2$ $\in f_2^{-1}\,\mathscr{M}_2$ since $f_2\,f_1^{-1}\,O_1 \cap A_2 \neq \phi$, $A_2 \in \mathscr{M}_2$. Therefore as ϱ satisfies R_4 and R_7 there is a ϱ-filter $\mathscr{F}$ containing $f_2^{-1}\,\mathscr{M}_2$ and the set $f_1^{-1}\,P_1$. If $\mathscr{M}$ is a maximal ϱ-filter containing $\mathscr{F}$ then $M(f_2)\mathscr{M}=\mathscr{M}_2$ and $f_1^{-1}\,P_1 \in \mathscr{M}$.

Corollary. $O_1\,\varrho_1\,P_1 \Rightarrow (X^n[O_1])^* \subset Y^n[P_1{}^*]$.

Proof: For $n=1$ it is an immediate consequence of the lemma and the fact that $X[O_1]=O_1+f_2\,f_1^{-1}\,O_1$ and $Y[P_1{}^*]=P_1{}^*+M(f_2)\;M(f_1)^{-1}P_1{}^*=$

$= P_1^* + M(f_2)\,(f_1^{-1} P_1)^*$ (since $M(f_1)\mathscr{M} = f_1\mathscr{M}$ it follows that $P_1 \in f_1\mathscr{M}$ iff $f_1^{-1} P_1 \in \mathscr{M}$).

Assume true for n and let $O_1\,\varrho_1\,Q_1\,\varrho_1\,P_1$. Then it is clear that $X^n[O_1]$ $(\varrho_1 + \varrho_2)\,X^n[Q_1]\,(\varrho_1 + \varrho_2)\,X^n[P_1]$. Furthermore for $n = 1$ the corollary clearly holds for $\varrho_1 + \varrho_2$. Hence $(X^{n+1}[O_1])^* \subset Y[(X^n[Q_1])^*] \subset Y[Y^n[P_1^*]] = Y^{n+1}[P_1^*]$.

Since $r[O] = X^n[O]$ for each open set O and some integer $n = n(O)$ it follows immediately from the corollary that $(r[O])^* \subset Y^n[P_1^*]$ for sufficiently large n. As $Y^n[P_1^*] \subset r_M[P_1^*]$ it follows that $(r[O_1])^* \subset r_M[P_1^*]$.

A particular case of the proposition is obtained when E is an open subspace of E_1 and f_1 is the inclusion. Then the pushout is denoted by $E_1 \cup_{f_2} E_2$. Let $\varrho_1 \cup_{f_2} \varrho_2$ be the pushout relation in this case. Hence if ϱ_1, ϱ_2 satisfy R_3, R_4 and R_7 and $f_1^{-1}(\varrho_1) = \varrho \subset \varrho_1$ and f_2 is open with $f_2(\varrho) \subset \varrho_2$ the following proposition holds.

Proposition 9. $M(\varrho_1 \cup_{f_2} \varrho_2) = M(\varrho_1) \cup_{M(f_2)} M(\varrho_2)$.

Proof: It is an immediate consequence of proposition 8 since $f_1 x = f_1 y$ implies $x = y$.

Note: This extends to n-fold attaching by induction since $\varrho_1 \cup_{f_2} \varrho_2$ satisfies R_3, R_4 and R_7 if each ϱ_i does.

§ 7. *A natural transformation*

For any object $(E, \mathscr{O}, \varrho)$ the functor M associates with a given set E a new set, the set $M(\varrho) = M(E, \mathscr{O}, \varrho)$. In many instances there is an obvious mapping of $E \to M(\varrho)$. This leads to the search for a natural transformation $m : F \to M$ where $F(E, \mathscr{O}, \varrho) = E$ and $F(f) = f$ i.e. F forgets all the structure and M is considered as a set-valued functor (or more precisely M is replaced by $F' \circ M$ where F' forgets topologies).

The following theorem and its corollary show that while such an m does not exist on the whole category it exists on a reasonably large subcategory.

Theorem 9. Let $m : F \to M$ be a natural transformation. Let $(E, \mathscr{O}, \varrho)$ be a triple such that:

(1) ϱ satisfies R_4; and

(2) for all $x \in E$ the largest ϱ-filter $\mathscr{F}_x$ contained in the neighborhood filter $\mathscr{V}(x)$ is a maximal ϱ-filter.

Then $m_\varrho x = \mathscr{F}_x$ for all $x \in E$. This formula defines a natural transformation of these functors restricted to the subcategory of triples satisfying (1) and (2).

Proof: Consider the trivial object consisting of the singleton $\{0\}$, the topology on $\{0\}$ and the containment relation ϱ_0. Then $\{\{0\}\}$ is the only ϱ_0-filter on $\{0\}$ and so $m_{\varrho_0} 0 = \{\{0\}\}$.

Let $(E, \mathcal{O}, \varrho)$ be a triple satisfying (1) and (2). Pick $x \in E$ and define $j_x = j : \{0\} \to E$ by $j0 = x$. Then j is an admissable map. Furthermore $j^{-1}\mathscr{F}_x = \{\{0\}\}$ and so $\{\{0\}\}\, M(j)\mathscr{F}_x$.

Since $\mathscr{F}_x$ is the only maximal ϱ-filter $\mathscr{M}$ for which $j^{-1}\mathscr{M}$ is proper it follows that $m_\varrho x = (m_\varrho \circ j)0 = (M(j) \circ m_{\varrho_0})\, 0 = \mathscr{F}_x$.

Consider the function $m_\varrho : E \to M(\varrho)$ defined by $m_\varrho x = \mathscr{F}_x$. If $(E', \mathcal{O}', \varrho')$ is another triple satisfying (1) and (2) and $f : E \to E'$ is an admissable map then $M(f) \circ m_\varrho = m_{\varrho'} \circ f$. This follows from the fact that $M(f)$ is a function and $f^{-1}\mathscr{F}'_{fx} \subset \mathscr{F}_x$ for each $x \in E$.

Corollary. There exists no natural transformation $m : F \to M$.

Proof: Assume m exists. Let $E = \mathsf{R}$ and let $\mathcal{O}$ be the usual topology. Consider the commutative diagram

$$\begin{array}{ccc} M(\gamma_4) & \xrightarrow{\quad M(e) \quad} & M(\gamma_1) \\ {\scriptstyle m_{\gamma_4}}\uparrow & & \uparrow{\scriptstyle m_{\gamma_1}} \\ \mathsf{R} & \xrightarrow{\quad e \quad} & \mathsf{R} \end{array}$$

Clearly $M(e) \circ m_{\gamma_4}$ is the void subset of $\mathsf{R} \times M(\gamma_1)$. This implies that m_{γ_1} is also the void subset of the same set. Since m_{γ_1} is a function on R this is a contradiction.

If f is a (ϱ, ϱ')-map and ϱ' satisfies (1) and (2) and $\varrho = f^{-1}(\varrho')$ the proof of theorem 4 shows that ϱ satisfies (1) and (2) since $f^{-1}\mathscr{V}(fx) \subset \mathscr{V}(x)$. Furthermore if $(\varrho_\alpha)_{\alpha \in \Omega}$ is a family of relations ϱ_α all satisfying (1) and (2) then it is clear that $\bigwedge_\alpha \varrho_\alpha$ satisfies these conditions and that the $\mathscr{F}_x$ are filters in $D(\bigwedge_\alpha \varrho_\alpha)$ i.e. $m_{\bigwedge_\alpha \varrho_\alpha} E \subset D(\bigwedge_\alpha \varrho_\alpha)$.

Consequently the relations $\varrho_i = \bigwedge_\alpha f_\alpha^{-1}(\gamma_{i(\alpha)})$, $i(\alpha) = 2, \pm 3$, or 4 and $\alpha \in \Omega$ all satisfy (1) and (2). The filters $\mathscr{F}_x$ in this case are the neighborhood filters for the weak topology defined by the family $(f_\alpha)_{\alpha \in \Omega}$ of functions $f_\alpha : E \to \mathsf{R}$.

Assume from now on that each relation satisfies conditions (1) and (2) of theorem 9 (unless otherwise stated) and that F and M denote the restrictions of F and M to the subcategory defined by these conditions.

The transformation $m : F \to M$ is a transformation of set-valued functors. By forgetting just the relation ϱ and not the topology the question arises as to whether m is still a transformation i.e. is each m_ϱ a continuous function?

Proposition 10. If ϱ satisfies R_7 then m_ϱ is continuous.

Proof: For any open set O, $m_\varrho^{-1}\, O^* = \bigcup \{P \mid \exists\, Q \text{ with } P \,\varrho\, Q \subset O\}$. Obviously $x \in m_\varrho^{-1}\, O^*$ iff $O \in \mathscr{F}_x$. Assume $x \in P$ and for some $Q \subset O$, $P \,\varrho\, Q$.

Then since ϱ satisfies R_4 and R_7, Q is in a ϱ-filter and so $O \in \mathscr{F}_x$. Conversely if $O \in \mathscr{F}_x$ there exist P and Q with $x \in P$ and $P \varrho Q \subset O$.

The proof of this proposition shows how ϱ associates with each open set O the subset $\hat{O} = \bigcup\{P \mid \exists\, Q \text{ with } P \varrho Q \subset O\}$. This correspondence defines a new relation $\hat{\varrho} = \{(\hat{O}, \hat{P}) \mid (O, P) \in \varrho\}$.

Proposition 11. For an arbitrary relation ϱ a filter $\mathscr{F}$ is a ϱ-filter iff it is a $\hat{\varrho}$-filter. Hence ϱ is equivalent to $\hat{\varrho}$. In addition if ϱ satisfies the conditions of theorem 9 and R_7, $m_\varrho^{-1}\, \varrho_M = \hat{\varrho}$ where $\varrho_M = \{(O^*, P^*) \mid O \varrho P\}$.

Proof: The first assertion is an immediate consequence of the fact that $O \varrho P \varrho Q$ implies $O \subset \hat{P} \hat{\varrho} \hat{Q} \subset Q$ and $\hat{O} \hat{\varrho} \hat{P} \hat{\varrho} \hat{Q}$ implies $\hat{O} \subset O \varrho P \subset \hat{Q}$.

Proposition 10 shows that $m^{-1} O^* = \hat{O}$ for each open set O. Hence $m_\varrho^{-1}\, \varrho_M = \hat{\varrho}$.

This proposition shows that m_ϱ is essentially a (ϱ, ϱ_M)-map and hence that $M(m_\varrho)$ is defined.

Theorem 10. Let ϱ satisfy R_7. Then $m_{\varrho_M} = M(m_\varrho) : M(\varrho) \to M(\varrho_M)$ is a homeomorphism. The maximal ϱ_M-filters are the neighborhood filters.

Proof: If $O^* \neq \phi$ then there exists $P \neq \phi$ with $P \varrho O$. Hence for $x \in P$, $O \in \mathscr{F}_x$ and so $O^* \cap m_\varrho E \neq \phi$. As $O^* \cap P^* = (O \cap P)^*$ this implies that $m_\varrho E$ is dense in $M(\varrho)$. Consequently $M(m_\varrho)$ is a homeomorphism by theorem 4 ($M(\varrho) = M(m_\varrho^{-1}(\varrho_M))$ and ϱ_M inherits R_4 from ϱ).

Obviously the neighborhood filters of $M(\varrho)$ are ϱ_M-filters. They are maximal if ϱ_M satisfies R_2. Assume that $O \varrho P$ and $P \notin \mathscr{M}$ i.e. $\mathscr{M} \notin P^*$. Then for some $Q \in \mathscr{M}$, $O \cap Q = \phi$ since ϱ satisfies R_4 and R_7. Since $O^* \cap Q^* = \phi$ this shows that the closure of O^* is contained in P^* i.e. ϱ_M satisfies R_2.

The function m_{ϱ_M} is defined and equals $M(m_\varrho)$ since $M(m_\varrho) \circ m_\varrho = m_{\varrho_M} \circ m_\varrho$.

Let $\mathfrak{M}$ be a maximal ϱ_M-filter and let $\mathscr{M} = m_\varrho^{-1} \mathfrak{M}$ ($m_\varrho^{-1} \mathfrak{M}$ is a maximal ϱ-filter by the proof of theorem 4). The filter $m_\varrho \mathscr{M}$ converges to $\mathscr{M}$ and so $\mathfrak{M} \subset m_\varrho \mathscr{M}$ generates a filter with $\mathscr{V}(\mathscr{M})$. This shows that $\mathfrak{M} = \mathscr{V}(\mathscr{M})$.

Let $(E, \mathscr{O}, \varrho)$ and $(E', \mathscr{O}', \varrho')$ be two triples and $f : E \to E'$ an admissable map. Assume that $M(f)$ is a continuous function with domain $M(\varrho)$. Since ϱ and ϱ' define the canonical relations ϱ_M and ϱ'_M it is natural to ask if $M(f)^{-1}(\varrho'_M) \subset \varrho_M$. This question can be answered by considering the sets $M(f)^{-1}(O')^*$, $O' \in \mathscr{O}'$.

It is clear that $M(f)^{-1}(O')^* \subset (f^{-1}O')^*$ since $O' \in \mathscr{M}'$ and $\mathscr{M} \supset f^{-1}\mathscr{M}'$ implies $f^{-1}O' \in \mathscr{M}$. Assume $P' \varrho' O'$. Then if ϱ' satisfies R_4 and R_7, $(f^{-1}P')^* \subset \subset M(f)^{-1}(O')^*$. Suppose $f^{-1} P' \in \mathscr{M}$. Then P' and $f\mathscr{M}$ generate a filter and so if $\mathscr{M}' \subset f\mathscr{M}$, $O' \in \mathscr{M}'$.

The arguement used to prove proposition 11 applies to prove

Proposition 12. Let f be an admissable map such that $M(f)$ is a continuous function with domain $M(\varrho)$. If ϱ' satisfies R_4 and R_7 then every $M(f)^{-1}(\varrho'_M)$-filter is a ϱ_M-filter.

Proof: From the above assertion it is clear that $M(f)^{-1}(\varrho'_M)$ is equivalent to the subrelation $(f^{-1}(\varrho'))_M$ of ϱ_M.

This result together with the previous theorem provide a proof of

Theorem 11. Let $(E, \mathcal{O}, \varrho)$ and $(E', \mathcal{O}', \varrho')$ be two triples and $f : E \to E'$ an admissable map such that:

(1) both relations satisfy R_4 and R_7;

(2) for both spaces the largest relation-filter contained in a neighborhood filter is maximal; and

(3) the domain of $M(f)$ is $M(\varrho)$.

Then $M(f)\, M(\varrho)$ is a closed subset of $M(\varrho')$.

Proof: Proposition 4 shows that $M(f)$ is a continuous function. Proposition 12 shows that if $\mathscr{F}'$ is a ϱ'_M-filter and $M(f)^{-1}\mathscr{F}'$ is a proper filter then it is a ϱ_M-filter.

Since ϱ and ϱ' both satisfy the conditions of theorem 10 the maximal ϱ_M-filters (ϱ'_M-filters) are the neighborhood filters.

Let $\mathscr{M}_0'$ be in the closure of $M(f)\, M(\varrho)$. Then $M(f)^{-1}\, \mathscr{V}'(\mathscr{M}_0')$ is a ϱ_M-filter. Let $\mathfrak{M}_0 = \mathscr{V}(\mathscr{M}_0)$ be a maximal ϱ_M-filter containing this filter. Then $M(f)\, \mathfrak{M}_0$ converges to $M(f)\, \mathscr{M}_0$. However $M(f)\, \mathfrak{M}_0 \supset M(f)[M(f)^{-1}\, \mathscr{V}'(\mathscr{M}_0')] \supset \mathscr{V}'(\mathscr{M}_0')$. Since $M(\varrho')$ is Hausdorff this implies that $M(f)\, \mathscr{M}_0 = \mathscr{M}_0'$.

§ 8. *Applications to realcompact spaces*

A completely regular space E will be said to be realcompact if it is homeomorphic to a closed subset of $\prod_{\alpha \in \Omega} \mathsf{R}$ for some index set Ω. Clearly the product of an arbitrary number of realcompact spaces is realcompact and a closed subset of a realcompact space is also realcompact.

The space R is realcompact and so are the subspaces R^+ and R^-. Any closed interval is realcompact. Consequently all the spaces $M(\gamma_i)$ $i = 2, \pm 3$ or 4 are all realcompact. Let $f : E \to \mathsf{R}$ be continuous. The spaces $M(f^{-1}(\gamma_i))$ are all realcompact as an immediate consequence of the corollary to theorem 4 and theorem 11.

Let $(E, \mathcal{O})$ be a topological space and let $(f_\alpha)_{\alpha \in \Omega}$ be a family of continuous real-valued functions on E. If $i : \Omega \to \{2, \pm 3, 4\}$ is a function let $\varrho_i = \bigwedge_\alpha f_\alpha^{-1}\, (\gamma_{i(\alpha)})$.

Theorem 12. $D(\varrho_i)$ is realcompact. For each α there is a unique continuous $\bar{\mathsf{R}}$-valued function $\bar{f}_\alpha$ on $D(\varrho_i)$ such that $\bar{f}_\alpha \circ m_{\varrho_i} = f_\alpha$. If $i(\alpha) = 2$ or f_α is bounded then $\bar{f}_\alpha$ is real-valued.

Proof: $D(\varrho_i)$ is homeomorphic under $h_\Omega \circ M(d_\Omega)$ to a closed subspace of $\prod_\alpha M(f_\alpha^{-1}\, (\gamma_{i(\alpha)}))$. Since each of the factors is realcompact the first assertion follows.

Identifying $M(\gamma_{i(\alpha)})$ with the appropriate subset of $\bar{\mathsf{R}}$ let $\bar{f}_\alpha = M(f_\alpha) \circ \circ \pi_\alpha \circ h_\Omega \circ M(d_\Omega)$ where π_α is the α-th. projection. Then $\bar{f}_\alpha \circ m_{\varrho_i} = M(f_\alpha) \circ \circ \pi_\alpha \circ h_\Omega \circ M(d_\Omega) \circ m \;\; = M(f_\alpha) \circ \pi_\alpha \circ h_\Omega \circ m_{\prod\limits_\alpha \varrho_\alpha} \circ d_\Omega = M(f_\alpha) \circ \pi_\alpha \circ \prod\limits_\alpha m_{\varrho_\alpha} \circ \circ d_\Omega = M(f_\alpha) \circ m_{\varrho_\alpha} = f_\alpha$ where $\varrho_\alpha = f_\alpha^{-1}(\gamma_{i(\alpha)})$.

The last assertion follows from the fact that $\bar{f}_\alpha \mathscr{M} = \lim\limits_{\mathscr{M}} f_\alpha$.

Corollary 1. $M(\mathop{\triangle}\limits_n \varrho_n)$ is realcompact and for each n there is a unique continuous $\bar{\mathsf{R}}$-valued function $\bar{f}_n$ on $M(\mathop{\triangle}\limits_n \varrho_n)$ such that $\bar{f}_n \circ m_{\mathop{\triangle}\limits_n \varrho_n} = f_n$. If $i(n) = 2$ or f_n is bounded then $\bar{f}_n$ is real-valued.

Proof: Since $M(e)^{-1} \circ m_{\mathop{\triangle}\limits_n \varrho_n} = m_{\mathop{\wedge}\limits_n \varrho_n}$ the result follows from the theorem and proposition 7.

Corollary 2. $M(\varrho_i)$ is realcompact if $\lim\limits_{\mathscr{M}} f_\alpha \in M(\gamma_{i(\alpha)})$ for each $\alpha \in \Omega$ and $\mathscr{M} \in M(\varrho_i)$. In particular this is the case if

$$i(\alpha) = \begin{cases} 2 \Rightarrow f_\alpha \text{ bounded} \\ +3 \Rightarrow f_\alpha \text{ bounded above} \\ -3 \Rightarrow f_\alpha \text{ bounded below.} \end{cases}$$

When this is so $M(\varrho_i)$ is compact.

Proof: The maximal $f^{-1}(\gamma_{i(\alpha)})$-filters are all of the form $f_\alpha^{-1}(\mathscr{V}(\lambda) \mid \mathsf{R})$, $\lambda \in M(\gamma_{i(\alpha)})$. Hence $\mathscr{M} \in D(\varrho_i)$ iff $\lim\limits_{\mathscr{M}} f_\alpha \in M(\gamma_{i(\alpha)})$ for each $\alpha \in \Omega$. The first assertion now follows immediately.

Let $\mathscr{M} \in M(\varrho_i)$ and let $\alpha \in \Omega$. Consider the filter $f_\alpha \mathscr{M}$. If $i(\alpha) = 2$ the filter has non-void adherence when f_α is bounded. Consequently $\mathscr{M}$ and $f^{-1}\mathscr{V}(\lambda)$ generate a filter for some $\lambda \in \mathsf{R}$. By maximality $\mathscr{M} \supset f_\alpha^{-1}\mathscr{V}(\lambda)$ and so $\lim\limits_{\mathscr{M}} f_\alpha = \lambda \in M(\gamma_2)$.

If $i(\alpha) = +3$ $\lim\limits_{\mathscr{M}} f_\alpha \in \mathsf{R}$ if adh $f_\alpha M \neq \phi$. When adh $f_\alpha M = \phi$ the fact that $f_\alpha \mathscr{M}$ is bounded above implies that $\lim\limits_{\mathscr{M}} f_\alpha = -\infty \in M(\gamma_{+3})$. Similarly if $i(\alpha) = -3$.

Let $4(\alpha) = 4$, $\forall \alpha \in \Omega$. Then if $i(\alpha)$ satisfies these conditions it is clear that $M(\varrho_i) = M(\varrho_4)$. Since $M(\gamma_4)$ is compact the result follows.

Corollary 3. $M(\varrho)$ is compact if ϱ satisfies R_2, R_4, R_5 and R_7. $C_{M(\varrho)} \circ m_\varrho$ is the uniform closure of the lattice generated by S^*. It coincides with S^* if in addition ϱ satisfies R_4.

Proof: The corollary to theorem 2 shows that ϱ is equivalent to $\bigwedge\limits_{f \in S^*} f^{-1}(\gamma_4)$. The first result follows from the previous corollary.

The theorem and the Stone–Weierstrass theorem prove the second since S^* separates the points of $M(\varrho)$. When ϱ satisfies R_4 then S^* is a uniformly closed lattice and so coincides with $C_{M(\varrho)} \circ m_\varrho$.

Comment. FREUDENTHAL [1] showed that $M(\varrho)$ was compact if ϱ satisfied R_2 to R_7 inclusive. This result shows that normal bases define compact spaces and that these spaces are extensions of the original space since the neighborhood filters are ϱ-filters for the relation defined by the base.

The relations determined by a normal basis satisfy R_2, R_3, R_4, R_5 and R_7. Consequently the algebra of continuous real-valued functions on the compact extension determined by the basis is isomorphic to the corresponding S^*. For the specific examples of normal bases the corresponding algebras S^* have the following explicit descriptions:

(1) for locally compact E and the normal basis of sets O with $\mathbf{\Gamma} O$ or $\complement O$ compact S^* is the uniform closure of the algebra generated by the constants and the functions with compact support;
(2) for zero-dimensional E and the normal basis of open-closed sets S^* is the uniform closure of the algebra generated by the continuous characteristic functions; and
(3) for rim compact E and the normal basis of rim compact open sets the algebra S^* is the algebra of functions f for which $\lambda < \mu \Rightarrow \exists$ rim compact O and P with $\{x \mid gx < \lambda\} \subset O \subset \mathbf{\Gamma} O \subset P \subset \{x \mid gx < \mu\}$, $g = \pm f$.

Note that in each case the algebra can be defined for an arbitrary topological space and that the weak topology defined by the algebra is the topology of the space iff it has the corresponding property. This suggests that the problem of compactification consists of two parts: first the selection of a specific Banach algebra of bounded continuous real-valued functions for a class of spaces; and second the characterization of those spaces for which the algebra determines the topology.

In the case of FREUDENTHAL's relation $\curlywedge$ [2] his maximal $\curlywedge$-compactification is the maximal ideal space of the algebra $A(\curlywedge)$ consisting of those bounded continuous real-valued functions f for which $\lambda < \mu \Rightarrow \{x \mid fx < \lambda\} \curlywedge \{x \mid fx > \mu\}$ [1]. The problem of characterizing those spaces which can be embedded in their maximal $\curlywedge$-compactification is not solved although GAL [9] has shown that they include the rim compact spaces.

Example 13. Let π be the polyhedral relation on a real topological vector space E (see example 6). Then $M(\pi)$ is compact. Furthermore it is clear that m_π embeds E in $M(\pi)$ if E is finite dimensional (assume Hausdorff). Presumably the converse holds. For R^n the space $M(\pi)$ is distinct from both R^n and $\beta \mathsf{R}^n$. The first assertion holds as $\prod^n \gamma_4$ and π are not equivalent. The second is an immediate consequence of the following

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[1]) They form a Banach algebra since the formal properties of $\curlywedge$ imply that $A(\curlywedge)$ is a uniformly closed lattice closed under multiplication by and addition of constants. Hence the arguement of theorem 3 applies.

observation: let $f(x_1, \ldots, x_n) = \sin \sqrt{\sum_{i-1}^{n} x_i^2}$; if $X \subset \mathsf{R}^n$ is unbounded and arcwise connected then $fX = [-1, 1]$. Since the non-convergent filters in $M(\pi)$ have a basis of such sets it is clear that f converges along none of them.

Proposition 13. $M(\varrho_i) = M(\varrho_4)$ for arbitrary i if $S = \{f_\alpha \mid \alpha \in \Omega\}$ has the following properties:

(1) $f \in S \Rightarrow f \wedge \lambda,\ f \vee \lambda \in S$ for $\lambda \in \mathsf{R}$;
(2) $f \in S \Rightarrow f + \lambda \in S,\ \lambda f \in S$;
(3) $f, g \in S \Rightarrow f + g \in S$;
(4) S is uniformly closed.

Proof: Let $\mathscr{M} \in M(\varrho_i)$ and let $\alpha \in \Omega$. Then $\lim\limits_{\mathscr{M}} f_\alpha \in \bar{\mathsf{R}}$.

By maximality $\lim\limits_{\mathscr{M}} f_\alpha = \infty$ or is in $\bar{\mathsf{R}}$. Assume $\lim\limits_{\mathscr{M}} f_\alpha = \infty$. Then the sets $A_n \cup B_n \in \mathscr{M}$ where $A_n = \{x \mid f_\alpha x < -n\}$ and $B_n = \{x \mid f_\alpha x > n\}$. Let $g_n = 1/2n\ [f_\alpha \wedge n) \vee (-n) + n]$. It is in S and so is $g = \sum_{n=1}^{\infty} g_n/2^n$. Since $0 \leqslant g \leqslant 1, \lim\limits_{\mathscr{M}} g \in [0, 1]$. Let $\lambda = \lim g$. If $0 < \lambda < 1$ then $\mathscr{M}$ is not a proper filter. If $\lambda = 0$ $\lim\limits_{\mathscr{M}} f_\alpha = -\infty$ and if $\lambda = 1$ $\lim\limits_{\mathscr{M}} f_\alpha = +\infty$.

Consequently $M(\varrho_i) \subset M(\varrho_4)$. Property (1) implies that the ϱ_4-filters are all determined by bounded functions in S. Hence every maximal ϱ_4-filter is a ϱ_i-filter and so $M(\varrho_4) \subset M(\varrho_i)$.

Corollary 1. Let C be the set of continuous real-valued functions on a completely regular space E. Let $\varrho_C = \bigwedge\limits_{f \in C} f^{-1}(\gamma_2)$. Then $(M(\varrho_C), m_{\varrho_C})$ is a realization of the Stone–Cech compactification βE of E. The same conclusion holds if C is replaced by C^* (the bounded functions in C) and γ_2 by any of the relations γ_i, $i = \pm 3$ or 4 (varying with f). The diagonal $(D(\varrho_C), m_{\varrho_C})$ is a realization of Hewitt's realcompactification vE.

Proof: The first assertion follows immediately from the proposition, theorem 12 and its second corollary.

The second assertion is a consequence of the proposition and the last assertion follows from theorem 12.

Corollary 2. Every compact space is of the form $M(\varrho)$ and every realcompact space is of the form $D(\bigwedge\limits_{\alpha} \varrho_\alpha)$.

Proof: E is compact (realcompact) iff $E = \beta E (E = vE)$.

Proposition 14. Let A be a closed subset of $\prod\limits_{n \in \mathbf{N}} E_n$. Let ϱ_n be a relation on E such that $m_{\varrho_n} : E_n \to M(\varrho_n)$ is a homeomorphism and (A_n, E_n, ϱ_n) satisfies (S) for all closed subsets A_n. Then $m_{\triangle_n \sigma_n} : A \to M(\triangle\limits_n \sigma_n)$ is a homeomorphism, where $\sigma_n = (\pi_n \mid A)^{-1} (\varrho_n)$.

Proof: Let $i : A \to \prod_n E_n$ be the inclusion mapping. Then $i = \prod_n i_n \circ \circ \prod_n (\pi_n \mid A) \circ d_{\mathsf{N}}$ where $A_n = \pi_n A$ and $i_n : A_n \to E_n$ is the inclusion. Since m_{ϱ_n} is defined $m_{\prod_n \varrho_n}$ is also defined and hence $m_{\triangle_n \sigma_n}$ is defined. The relation $M(i) = \prod_n M(i_n) \circ \prod_n M(\pi_n \mid A) \circ h_{\mathsf{N}} \circ M(d_{\mathsf{N}})$ and it is an embedding since each of the factors is an embedding or a homeomorphism. The map $\prod_n m_{\varrho_n}$ is a homeomorphism and so $(\prod_n m_{\varrho_n})^{-1} \circ M(i)$ embeds $M(\triangle_n \sigma_n)$ in $\prod_n E_n$. A is closed and since $m_{\triangle_n \sigma_n}$ embeds it as a dense subset of the subspace $M(\triangle_n \sigma_n)$ of $\prod_n E_n$ it follows that $m_{\triangle_n \varrho_n}$ is a homeomorphism.

Corollary. Every closed subset of $\prod_n \mathsf{R}$ is of the form $M(\varrho)$.

Proof: R and γ_2 satisfy the conditions of the proposition.

Problem. The last two corollaries show that the spaces $M(\varrho)$ form a fairly large class which by example 8 also includes all discrete spaces. This raises the problem of characterizing those spaces which are $M(\varrho)$ for a suitable relation: in particular is every realcompact space (or more generally is every complete uniform space of this form). A more restricted problem is that of characterizing those spaces E for which there is a relation ϱ on the topology such that m_ϱ is defined and is a homeomorphism of E with $M(\varrho)$. Theorem 10 states that $M(\varrho)$ has this property if ϱ satisfies R_4, R_7 and each $\mathscr{F}_x$ is maximal. It is not clear whether the resticted problem coincides with the original one.

To conclude this section consider the following duality between relations ϱ satisfying R_1 to R_8 inclusive and Banach subalgebras A of C^*. For such a relation S^* is a Banach subalgebra of C^* (theorem 3). Denote it by $A(\varrho)$. Then by corollary 3 of theorem 12 $A(\varrho) = C_{M(\varrho)} \circ m_\varrho$.

On the other hand a Banach subalgebra A of C^* defines the relation $[\bigwedge_{f \in A} f^{-1}(\gamma_4)]^{\dot{-}}$ which satisfies R_1 to R_8 inclusive. Let it be denoted by $\varrho(A)$.

Theorem 13. $A(\varrho\,(A)) = A$ and $\varrho\,(A(\varrho)) = \varrho$.

Proof: The corollary to theorem 1 implies that $\varrho(A(\varrho)) = \varrho$.

The algebra $A(\varrho\,(A)) = C_{M(\varrho(A))} \circ m_{\varrho(A)}$ which contains A since $\varrho(A)$ is equivalent to $\bigwedge_{f \in A} f^{-1}(\gamma_4)$. Since the subalgebra $\bar{A} = \{\bar{f} \mid f \in A\}$ of $C_{M(\varrho(A))}$ separates the points, the Stone–Weierstrass theorem shows that $\bar{A} = C_{M(\varrho(A))}$. The result follows since $\bar{f} \circ m_{\varrho(A)} = f$.

Corollary. There is a $1-1$ correspondence between the relations ϱ satisfying R_1 to R_8 inclusive and the equivalence classes of continuous maps $\varphi : E \to K$ where K is compact and φE is dense in K. The correspondence associates m_ϱ with ϱ.

Proof: Obviously m_ϱ is such a map. Furthermore m_{ϱ_1} is equivalent to m_{ϱ_2} (i.e. there is a homeomorphism $m_{21} : M(\varrho_1) \to M(\varrho_2)$ with $m_{21} \circ m_{\varrho_1} = m_{\varrho_2}$) iff $\varrho_1(A) = \varrho_2(A)$ i.e. iff $\varrho_1 = \varrho_2$.

Let φ be a map from E to K. Let $A = C_K \circ \varphi$. It is isomorphic to C_K as φE is dense in K. $(M(\varrho(A)), m_{\varrho(A)})$ is equivalent to (K, φ) in view of theorem 4 and the fact that $m_{\varrho_{C_K}}$ is a homeomorphism.

§ 9. *The cut completion of a chain*

Unless measurable cardinals are assumed to exist all the discrete spaces are realcompact [10]. Consequently none of the examples given of the spaces $M(\varrho)$ is demonstrably not realcompact. The simplest examples of spaces that are not realcompact are the non-compact pseudocompact spaces – for example the space $W(\omega_1)$ of ordinals less then the first uncountable ordinal ω_1. The underlying set is a chain (i.e. totally ordered) and conditionally complete. The argument used to compute $M(\gamma_2)$ will now be adapted to any chain to prove that the conditionally complete chains are all of the form $M(\varrho)$.

Let $E = (E, \leqslant)$ be a chain and let $\mathcal{O}$ be the interval topology i.e. the topology generated by the sets $(\leftarrow, a) = \{x \mid x < a\}$ and $(a, \rightarrow) = \{x \mid x > a\}$. Let ϱ be the relation γ restricted to the bounded intervals that are open subsets i.e. the intervals (a, b) together with the intervals $[0, b)$ if E has a least element 0 and $(a, 1]$ if E has a largest element 1.

If $x \in E$ then $\mathscr{V}(x)$ is a maximal ϱ-filter. Let U be a neighborhood of x. Assume $x \neq 0$ or 1. Then U contains an interval (a, b) containing x. If there exist c, d with $a < c < x < d < b$ then obviously $(c, d)\ \varrho\ (a, b)$. If c exists and d not then $(a, b) = (a, x]$ and $(c, b) = (c, x]\ \varrho\ (a, x]$. Similarly if d exists and c does not $(a, d)\ \varrho\ (a, b)$. If neither c nor d exist then $(a, b) = [x, x] = \{x\}$ and $(a, b)\ \varrho\ (a, b)$. Since ϱ satisfies R_2 $\mathscr{V}(x)$ is a maximal ϱ-filter. The arguement for $x = 0$ or 1 is the same.

The relation ϱ satisfies R_2, R_4 and R_7. It satisfies R_2 by definition and clearly satisfies R_4 since the intersection of two bounded intervals is a bounded interval. Assume $(c, d)\ \varrho\ (a, b)$. Then $a \leqslant c < d \leqslant b$. If e, f exist with $a < e < c$ and $d < f < b$ then $(c, d)\ \varrho\ (e, f)\ \varrho\ (a, b)$. If e exists but f does not then $(a, b) = (a, d]$ and $(c, d)\ \varrho\ (e, d]\ \varrho\ (a, d]$. Similarly if f exists but e does not, an interval can be found. Also the same arguements apply when $c = a = 0$ and $[0, d)\ \varrho\ [0, b)$ or when $d = b = 1$ and $(c, 1]\ \varrho\ (a, 1]$. This shows that ϱ satisfies R_7. Consequently theorem 1 shows that the interval topology is completely regular.

A *cut* of E is an ordered pair $[A, B]$ of subsets A, B of E such that:

(1) $A \neq \phi$, $B \neq \phi$;

(2) $a \in A$, $b \in B \Rightarrow a \leqslant b$; and

(3) $[A, B]$ is maximal with respect to (1). If $[A, B]$ is not a point cut i.e. sup A and inf B do not exist then the intervals (a, b) $a \in A$ and $b \in B$ generate a non-convergent ϱ-filter. This follows immediately from the fact that $a \in A$, $b \in B \Rightarrow \exists\ a' \in A$, $b' \in B$ with $a < a' < b' < b$. The filter

is maximal since $[c, d] \cap (a, b) \neq \phi \ \forall\ a \in A, b \in B \Rightarrow A \cup \{c\}$ and $B \cup \{d\}$ satisfy (1).

Conversely if $\mathscr{M}$ is any non-convergent maximal ϱ-filter and $A_{\mathscr{M}} = \{a \mid [a, x] \in \mathscr{M}$ for some $x\}$, $B_{\mathscr{M}} = \{b \mid [y, b] \in \mathscr{M}$ for some $y\}$ then $[A_{\mathscr{M}}, B_{\mathscr{M}}]$ is a cut which is not a point cut and whose associated maximal ϱ-filter is $\mathscr{M}$. Clearly the cut defined by the maximal ϱ-filter associated with $[A, B]$ is $[A, B]$.

Let $C = C(\leqslant)$ denote the set of cuts of E and let $c : E \rightarrow C$ be the function defined by $cx = [(\leftarrow, x], [x, \rightarrow)]$. Define the function $n : M(\varrho) \rightarrow C$ by setting

$$n\mathscr{M} = \begin{cases} cx \text{ if } \mathscr{M} = \mathscr{V}(x) \\ [A_{\mathscr{M}}, B_{\mathscr{M}}] \text{ otherwise.} \end{cases}$$

Then obviously $n \circ m_{\varrho} = c$ and n is a bijection.

The cuts of E can be totally ordered by setting $[A_1, B_1] \leqslant [A_2, B_2]$ if $A_1 \subset A_2$. Then c is an order preserving injection. Furthermore the order on C defines an interval topology.

Theorem 14. The following assertions hold:

(1) ϱ_M is equivalent to the relation defined by the order of C;
(2) n is a homeomorphism;
(3) $(E, \leqslant)$ is conditionally complete iff m_{ϱ} is a homeomorphism; and
(4) C is conditionally complete.

Proof: The relation ϱ satisfies the conditions of theorem 10 and so the neighborhood filters of $M(\varrho)$ are the maximal ϱ_M-filters. Consequently (1) implies (2).

$(E, \leqslant)$ is conditionally complete iff each cut is a point cut. Since c is an embedding this is so iff c is a homeomorphism. Hence (2) implies (3).

(4) follows from (3) by theorem 10. It remains therefore to prove (1).

Let $\leqslant$ also denote the order of C carried over to $M(\varrho)$ by n. Then $\mathscr{M}_1 \leqslant \mathscr{M}_2$ iff $A_{\mathscr{M}_1} \subset A_{\mathscr{M}_2}$ where $A_{\mathscr{V}(x)} = (\leftarrow, x]$.

$\mathscr{M}_1 < \mathscr{M}_2 \Rightarrow \exists\, x < y$ in E with $\mathscr{M}_1 \leqslant \mathscr{V}(x) < \mathscr{V}(y) \leqslant \mathscr{M}_2$. $A_{\mathscr{M}_1} \nsubseteq A_{\mathscr{M}_2}$ and since one of the $\mathscr{M}_i$ – say $\mathscr{M}_1$ – can be assumed to be nonconvergent there exist $x < y$ in $A_{\mathscr{M}_2}$ with $A_{\mathscr{M}_1} < x$ ($A_{\mathscr{M}_1}$ has no supremum). Obviously $\mathscr{M}_1 \leqslant \mathscr{V}(x) < \mathscr{V}(y) \leqslant \mathscr{M}_2$. If $A_{\mathscr{M}_2}$ is assumed to be nonconvergent such x, y can again be found as $A_{\mathscr{M}_2}$ has no supremum.

If $\mathscr{M}$ has a successor (predecessor) then $\mathscr{M}$ converges. $\mathscr{V}(d)$ precedes $\mathscr{V}(b)$ iff d precedes b. The first assertion is now obvious. The second is obvious as c is order preserving.

Obviously $\mathbf{\Gamma}(\leftarrow, a) = (\leftarrow, a)$ iff a has a predecessor. $\mathbf{\Gamma}(\leftarrow, d) \subset (\leftarrow, b)$ iff $\mathbf{\Gamma}(\leftarrow, \mathscr{V}(d)) \subset (\leftarrow, \mathscr{V}(b))$. This is clear if either d or $\mathscr{V}(d)$ has a predecessor. If this is not the case then $d < b$ and the result follows. Clearly $\mathbf{\Gamma}(a, b) = \mathbf{\Gamma}(a, \rightarrow) \cap \mathbf{\Gamma}(\leftarrow, b)$ if $(a, b) \neq \phi$.

Let $\tilde{\varrho}$ denote the relation defined by the order $\leqslant$. Then if $(\mathscr{M}_3, \mathscr{M}_4)$

$\tilde{\varrho}(\mathscr{M}_1, \mathscr{M}_2)$ with $\mathscr{M}_3 \neq \mathscr{M}_4$, there exist a, b, c, d in E with $(\mathscr{M}_3, \mathscr{M}_4) \subset (\mathscr{V}(c), \mathscr{V}(d))\, \tilde{\varrho}\, (\mathscr{V}(a), \mathscr{V}(b)) \subset (\mathscr{M}_1, \mathscr{M}_2)$. This follows immediately from the observations about closure and the consequence of $\mathscr{M}_4 < \mathscr{M}_2$, $\mathscr{M}_1 < \mathscr{M}_3$. Similarly if $0 \in E$ and $[\mathscr{V}(0), \mathscr{M}_4)\, \tilde{\varrho}\, [\mathscr{V}(0), \mathscr{M}_2)$ there exist, b, d in E with $[\mathscr{V}(0), \mathscr{M}_4) \subset [\mathscr{V}(0), \mathscr{V}(d))\, \tilde{\varrho}\, [\mathscr{V}(0), \mathscr{V}(b)) \subset [\mathscr{V}(0), \mathscr{M}_2)$. The corresponding assertion in case $1 \in E$ clearly also holds.

For any $a \in E$ $(\leftarrow, a)^* = (\leftarrow, \mathscr{V}(a))$ and $(\mathscr{V}(a), \rightarrow) = (a, \rightarrow)^*$. If $(\leftarrow, a) \in \mathscr{M}$ then $a \in B_{\mathscr{M}}$. When $\mathscr{M}$ is nonconvergent this implies $a > A_{\mathscr{M}}$ and so $\mathscr{M} \in (\leftarrow, \mathscr{V}(a))$. If $\mathscr{M}$ converges it does so to a point $c < a$. Conversely if $A_{\mathscr{M}} \notin (\leftarrow, a]$ and $\mathscr{M}$ does not converge there exists $c < a$ with $A_{\mathscr{M}} < c$ and so $(\leftarrow, c] \in \mathscr{M}$. As $(\leftarrow, a) \supset (\leftarrow, c]$ $\mathscr{M} \in (\leftarrow, a)^*$. For a convergent $\mathscr{M} = \mathscr{V}(c)$ $c < a$ and so $(\leftarrow, a) \in \mathscr{V}(c)$. This proves the first assertion. The second follows immediately using the opposite order.

$$(c, d)^*\ \varrho_M (a, b)^* \Leftrightarrow (\mathscr{V}(c), \mathscr{V}(d))\, \tilde{\varrho}\, (\mathscr{V}(a), \mathscr{V}(b))$$

if $(c, d) \neq \phi$. This follows from the last paragraph and the observations on closure. Similar statements hold in case $0 \in E$ $(1 \in E)$ and for intervals $[0, d)^*$ and $[0, b)^*$ ($(c, 1]^*$ and $(a, 1]^*$). Consequently a filter $\mathscr{F}$ is a ϱ_M-filter iff it is a $\tilde{\varrho}$-filter and so ϱ_M is equivalent to $\tilde{\varrho}$.

Note. It appears possible that this theorem will also hold for lattices. It does not work for an arbitrary partially ordered set. Consider two intersecting directed lines in the plane with the point of intersection removed. The cut completion of the resulting partially ordered set is obtained by replacing the point of intersection. However the filter construction adds four points.

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REFERENCES

1. Freudenthal, H., Neuaufbau der Endentheorie, Ann. of Math., 43, 261–279 (1942).
2. ———, Bündige Räume, Fund. Math., 48, 307–312 (1960).
3. ———, Uber die Enden topologischer Räume und Gruppen, Math. Zeitschrift, 33, 692–713 (1931).
4. Császár, Á., Fondements de la topologie générale, Budapest 1960.
5. Fan, K., and N. Gottesman, On the compactifications of Freudenthal and Wallman, Indagationes Math., 14, 504–510 (1952).
6. Carathéodory, C., Uber die Begrenzung einfach zusammenhangender Gebiete, Math. Ann., 73, 323–370 (1913).
7. Groot, J. de, Sätze über topologische Erweiterung von Abbildungen, Indagationes Math., 3, 419–424 (1941).
8. Smirnov, Y., On proximity spaces, Mat. Sbornik N.S., 31, 543–574 (1952).
9. Gál, I. S., Proximity relations and precompact structures, Indagationes Math., 22, 304–326 (1959).
10. Gillman, L. and M. Jerison, Rings of continuous functions, New York, 1960.
11. Halpern, J. D., Some structure in the hierarchy of maximal principles in set theory, Notices Amer. Math. Soc. 9 # 4 Abstract 62T–178, (1962).